Rational Hyperplane Arrangements and Counting Independent Sets of Symmetric Graphs

MIT PRIMES Conference

Nicholas Guo

Mentor: Guangyi Yue

May 21, 2016

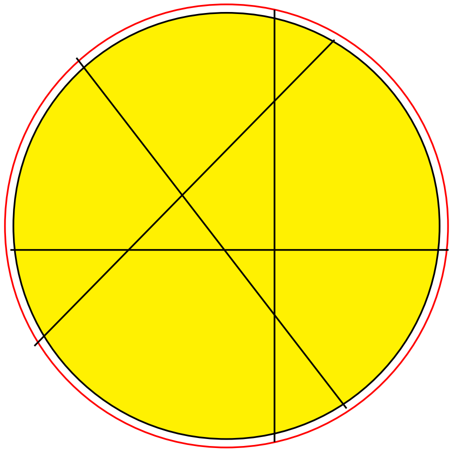
Hyperplane Arrangements

Definition

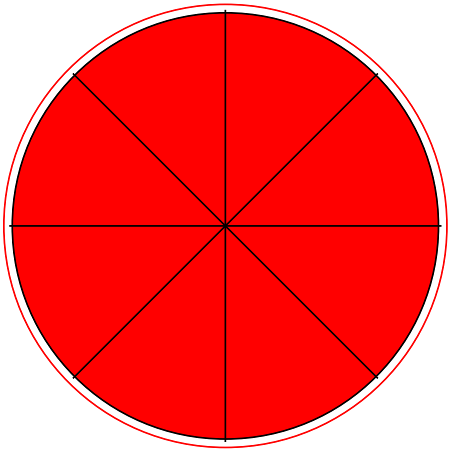
- Begin with the Euclidean space $\mathbb{R}^n = \{(x_1, x_2, ..., x_n) : x_i \in \mathbb{R}\}$.
- An affine hyperplane is the set of points in $\mathbb{R}^n$ satisfying an equation of the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$.
- A hyperplane arrangement is simply a finite collection of affine hyperplanes.
- A hyperplane arrangement is central if all of its hyperplanes pass through at least one point.

Familiar Examples in $\mathbb{R}^2$

Hyperplanes are lines in $\mathbb{R}^2$



11 pieces

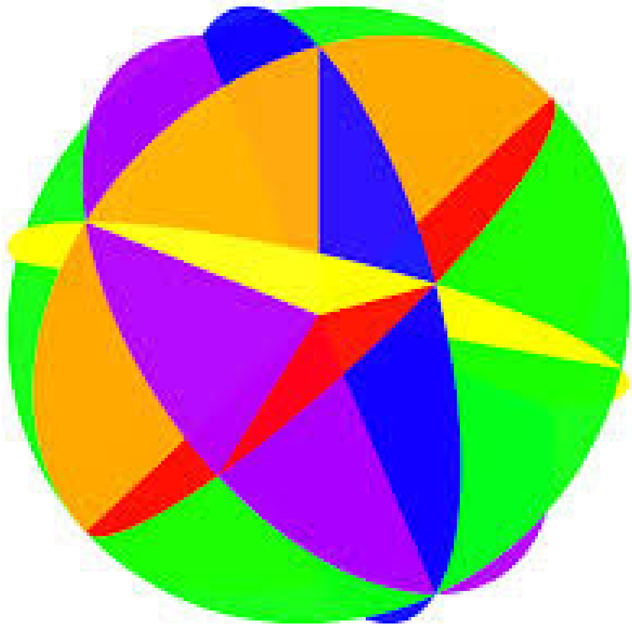


8 pieces

Cake cut for maximum # pieces
Hyperplane arrangement in a general position

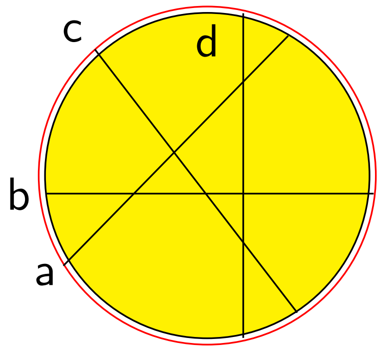
Pizza cut through a center
A central hyperplane arrangement

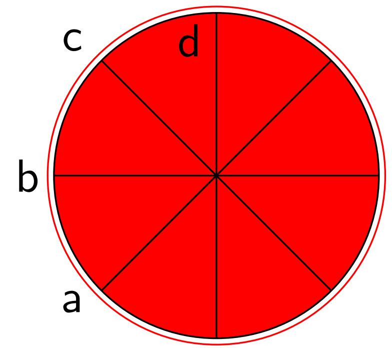
An Example in $\mathbb{R}^3$

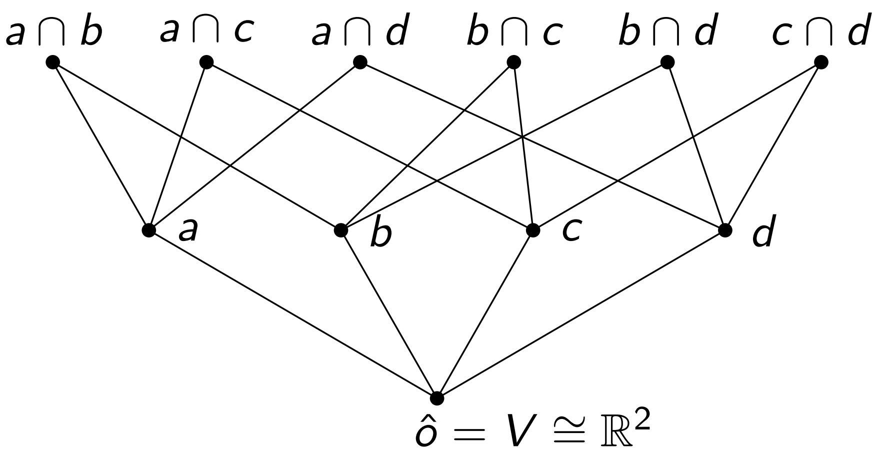


A central hyperplane arrangement of 6 planes in a 3-dimensional space.

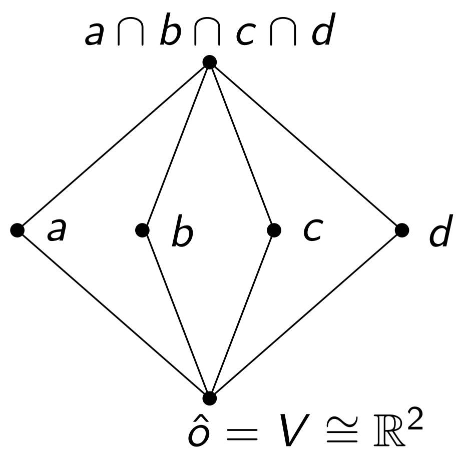
An Arrangement and Its Intersection Poset







Poset for Cake cut



Poset for Pizza cut

The Möbius Function for Posets

Definition

Let P be a locally finite poset. Define a function $\mu = \mu_P : \text{Int}(P) \to \mathbb{Z}$, called the Möbius function of P, by the conditions:

$$\mu(x, x) = 1, \forall\ x \text{ in } P$$

$$\mu(x, y) = -\sum_{x \leq z < y} \mu(x, z), \ \forall\ x < y \text{ in } P.$$

- The second condition can be be written:

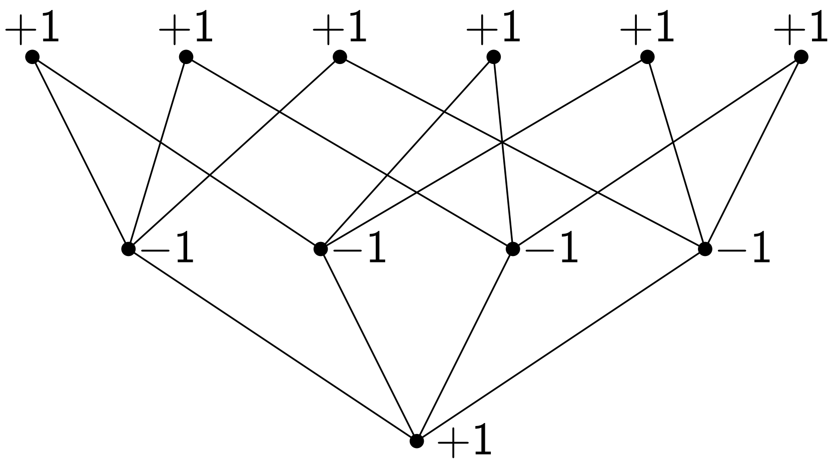
$$\sum_{x \leq z \leq y} \mu(x, z) = 0, \ \forall\ x < y \text{ in } P.$$

Möbius Function Values and the Characteristic Polynomial

Definition

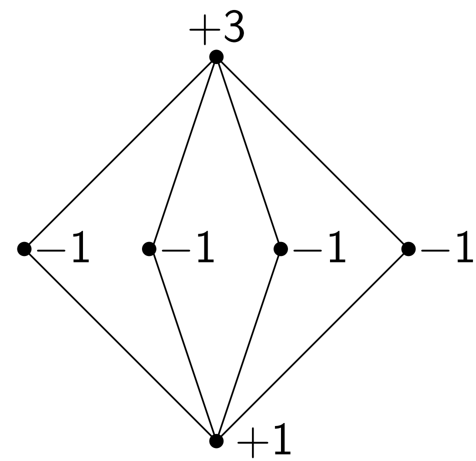
The characteristic polynomial of a hyperplane arrangement is defined by:

$$\chi_{\mathcal{A}}(t) = \sum_{x \in L(\mathcal{A})} \mu(\hat{0}, x) t^{\dim(x)}$$



$\chi_{\mathcal{A}}(t) = t^2 - 4t + 6$

"Cake-cut" arrangement



$\chi_{\mathcal{A}}(t) = t^2 - 4t + 3$

"Pizza-cut" arrangement

Regions

Definition

- A region of the arrangement $\mathcal{A}$ is a connected component of the complement

$$\mathbb{R}^n - \cup_{H\in\mathcal{A}} H.$$

- $r(\mathcal{A})$ denotes the total number of regions, and $b(\mathcal{A})$ denotes the number of bounded regions.

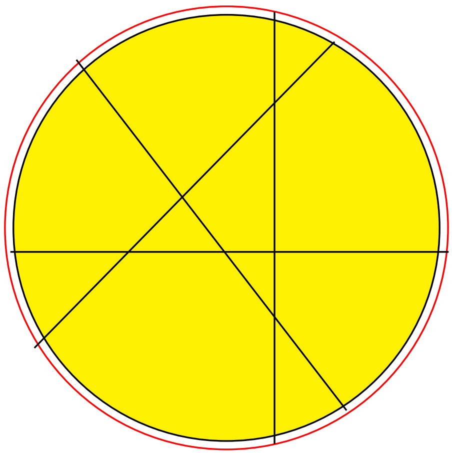
Theorem (Zaslavsky)

The number of regions and bounded regions can be found as:

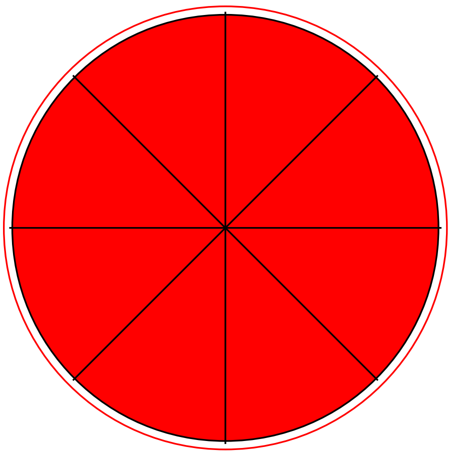
$$r(\mathcal{A}) = |\chi_{\mathcal{A}}(-1)|$$

$$b(\mathcal{A}) = |\chi_{\mathcal{A}}(1)|$$

Region Counting Again!



11 pieces



8 pieces

$$\chi_{\mathcal{A}}(t) = t^2 - 4t + 6$$
$$r(\mathcal{A}) = (-1)^2 - 4(-1) + 6 = \mathbf{11}$$
$$b(\mathcal{A}) = 1^2 - 4(1) + 6 = \mathbf{3}$$

$$\chi_{\mathcal{A}}(t) = t^2 - 4t + 3$$
$$r(\mathcal{A}) = (-1)^2 - 4(-1) + 3 = \mathbf{8}$$
$$b(\mathcal{A}) = 1^2 - 4(1) + 3 = \mathbf{0}$$

Another Example of a Hyperplane Arrangement

Let $\mathcal{A}_n$ be the arrangement in $\mathbb{R}^n$ with hyperplanes

$$\begin{aligned} x_i &= 0 & \forall\ i \\ x_i &= x_j & \forall\ i < j \\ x_i &= 2x_j & \forall\ i \neq j \\ x_i &= 3x_j & \forall\ i \neq j \end{aligned}$$

Find $\chi_{\mathcal{A}_n}(t)$.

$$\begin{aligned} \chi_{\mathcal{A}_2}(t) &= (t-1)(t-6) \\ \chi_{\mathcal{A}_3}(t) &= (t-1)(t^2-17t+78) \\ \chi_{\mathcal{A}_4}(t) &= (t-1)(t^3-33t^2+386t-1608) \\ \chi_{\mathcal{A}_5}(t) &= (t-1)(t^4-54t^3+1151t^2-11514t+45840) \\ &\dots \end{aligned}$$

Calculating $\chi_{\mathcal{A}_n}(t)$ becomes more difficult for higher dimensions.

The Finite Field Method

The characteristic polynomial of a Rational Arrangement can be found alternatively:

Theorem

Let $\mathcal{A}$ be any subspace arrangement in $\mathbb{R}^n$ defined over the integers and q be a large enough prime number, then:

$$\chi_{\mathcal{A}}(q) = \#(\mathbb{F}_q^n - \cup_{H \in \mathcal{A}} H) = q^n - \cup_{H \in \mathcal{A}} H.$$

Equivalently, identifying $\mathbb{F}_q^n$ with $[0, 1, ..., q-1]^n$,

$$\chi_{\mathcal{A}}(q) = \# \text{ of points with integer coordinates in } [0, q-1]^n$$

which do not satisfy mod q the defining equations of any of the subspaces in $\mathcal{A}$.

The Finite Field Method : An Example

For the hyperplane arrangement in $\mathbb{R}^n$

$$x_i = 0 \qquad \forall\, i$$
$$x_i = x_j \qquad \forall\, i < j$$
$$x_i = 2x_j \qquad \forall\, i \neq j$$

		x_1										
		0	1	2	3	4	5	6	7	8	9	10
x_2	0	X	X	X	X	X	X	X	X	X	X	X
	1	X	X	X				X				
	2	X	X	X		X						
	3	X			X			X	X			
	4	X		X		X				X		
	5	X					X			X		X
	6	X	X		X			X				
	7	X			X				X		X	
	8	X				X	X			X		
	9	X							X		X	X
	10	X					X				X	X

$$\chi_{\mathcal{A}}(p) = (p-1)(p-n-2)_{n-1} \qquad \mathbb{F}_p^2 = [0, p-1]^2,\ p = 11$$

where $(x)_m = x(x-1)\cdots(x-m-1)$.

Applications of Hyperplane Arrangements

Hyperplane arrangements have increasing applications in:

- biology,
- mathematical physics,
- statistical economics,
- topology of collision-free robot motion planning,
- machine learning and deep neural networks for Artificial Intelligence,
- combinatorics and graph theory,

...

Independent Sets of $G(V, E)$

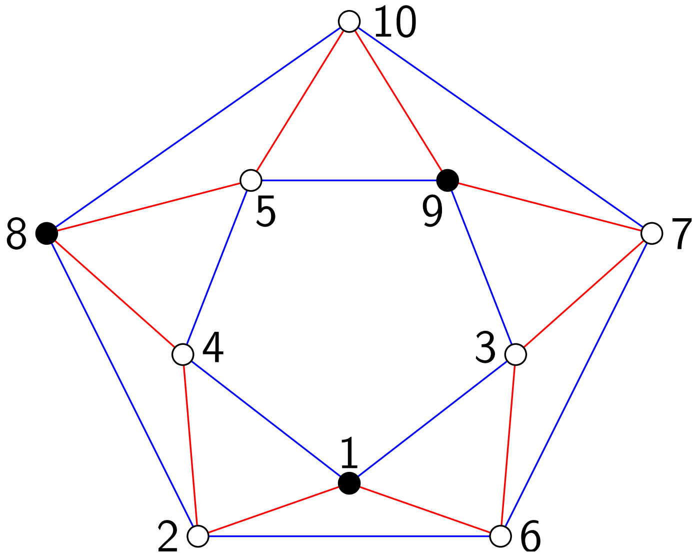
Definition

In a graph, an independent set is a set of vertices, no two of which are adjacent, or connected by an edge.

Example:

Hyperplane arrangement $\mathcal{A}_n$ in $\mathbb{R}^n$

$$x_i = 0 \qquad \forall\, i$$
$$x_i = x_j \qquad \forall\, i < j$$
$$x_i = 2x_j \qquad \forall\, i \neq j$$
$$x_i = 3x_j \qquad \forall\, i \neq j$$



How many 3-element independent sets of G on vertices [10] with edges ij: $j = 2i$ mod 11 (red line) and $j = 3i$ mod 11 (blue line)?

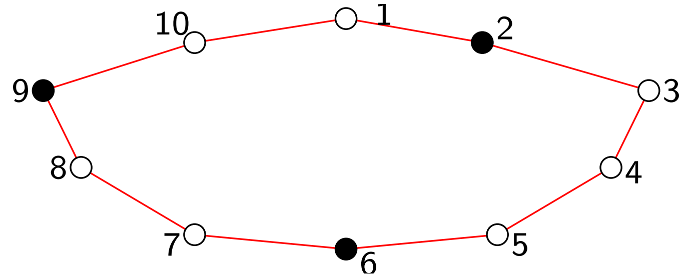
Counting Independent Sets of Symmetric Graphs

A simple problem

Choose n labeled points from a circular arrangement of $p-1$ points (cycle graph C_{p-1}). What is the number of n-element independent sets?

Example: C_{10}

$$\begin{aligned} x_i &= 0 & \forall\, i \\ x_i &= x_j & \forall\, i < j \\ x_i &= x_j + 1 & \forall\, i \neq j \end{aligned}$$



Solution

The number of n-element independent sets is the characteristic polynomial:

$$\chi_n(p) = (p-1)(p-n-2)_{n-1}$$

where $(x)_m = x(x-1)\cdots(x-m-1)$.

Independent Sets of Graph with Disjoint Union of Orbits

Theorem (Prior Conjecture)

Let $a = \{a_1, a_2, ..., a_m\}$ be a set of coprime integers. For an integer $k \gg 1$, let $G(k)$ be the graph with vertex set $\mathbb{Z}/k\mathbb{Z}$ and edges ij if $i \equiv a_r j \mod (k+1)$ for some r. Let G be the disjoint union $G(n_1) \cup G(n_2) \cup \cdots \cup G(n_s)$ ($n_1 + 1, n_2 + 1, ..., n_s + 1$ all primes $\gg 1$), then the number of n-element independent sets of G depends only on n, m, and $\sum n_i$.

Invariance of Characteristic Polynomial

Lemma

Let $\mathcal{A}_n$ be the arrangement in $\mathbb{R}^n$ with hyperplanes

$$\begin{aligned}
x_i &= 0 && \forall\, i \\
x_i &= x_j && \forall\, i < j \\
x_i &= a_1 x_j && \forall\, i \neq j \\
x_i &= a_2 x_j && \forall\, i \neq j \\
&\dots && \\
x_i &= a_m x_j && \forall\, i \neq j
\end{aligned}$$

For a fixed m, $\chi_{\mathcal{A}_n}(t)$ is independent of a_i's as long as they are coprime.

- We proved this by using generating functions.

Graph with Disjoint Union of Orbits

Theorem

Let $a = \{a_1, a_2, ..., a_k\}$ be a set of positive integers. For an integer $n \gg 1$, Let $G(a, n)$ be the graph with vertex set $\mathbb{Z}/n\mathbb{Z}$ and edges ij if $i - j \equiv a_r \mod n$ for some r. For some $n_1, n_2, ..., n_s \gg 0$, let G be the disjoint union $G(a, n_1) + G(a, n_2) + \cdots + G(a, n_s)$. Then the number of l-element independent sets of G depends only on a, l, and $\sum n_i$.

- We proved this similarly by employing its corresponding characteristic polynomial of hyperplane arrangements.

Future Research

- Generalization:

  $G = G_{n_1} + G_{n_2} + \cdots + G_{n_k}$ or $\mathbb{Z}/n_1\mathbb{Z} \cup \mathbb{Z}/n_2\mathbb{Z} \cup \cdots \cup \mathbb{Z}/n_k\mathbb{Z}$ be a graph which is the disjoint union of k graphs G_{n_i} which has n_i vertices. What kind of such graph has the property that the number of n-element independent sets depends solely on n and $\sum_i n_i$?

Acknowledgments

Many thanks to :

- Guangyi Yue, my mentor
- Prof. Richard P. Stanley for suggesting this project
- Dr. Tanya Khovanova, Prof. Pavel Etingof and Dr. Slava Gerovitch
- My parents
- MIT PRIMES USA